Education, Research, Innovation
ERIN 2008

Model Predictive Control in Vibration Attenuation

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1 INTRODUCTION

This article discusses the potential of using model predictive control algorithms in vibration control applications. In addition to the requirements arising from the high sampling rate nature of vibration attenuation systems, issues regarding guaranteed stability and constraint handling are discussed.

Vibration is present in countless real life applications, and most of the time it is a highly undesirable phenomenon. Unwanted vibration may decrease product performance, cause economic or safety problems. Engineers and scientists are constantly working to solve this complex question. All physical systems have some inherent physical damping, but this level may not be satisfactory in all cases. To increase energy dissipation one may apply vibration attenuation techniques. This can be carried out by passive means, taking advantage of the physical properties of the system. Due to several issues, passive treatments are not always viable.

Active vibration damping is an attractive alternative. This method employs actuators to utilize force effect on the system in question. The actuators are driven by control systems which gain feedback from one to several sensors. Active vibration attenuation systems are usually highly integrated and may be regarded as a complex mechatronic unit. Current promising applications include vibration damping in space structures, commercial and military aircraft, satellites, medical equipment, precision manufacturing and many more.

There is one important but often overlooked aspect of active vibration attenuation systems: the control algorithm. The obvious and the simplest choices are already well investigated. Basic controller implementations do not always provide the necessary performance. In some cases numerical stability is also questionable. Input and output constraints may be required because of actuator limitations, safety or economic considerations.

Currently the only control technique which is capable of handling constraints while considering their effects on future control moves is model predictive control. [1] (MPC) Predictive control has been around for quite some time and has been successfully implemented in applications, where sampling rates were in the order of tens of minutes. Thanks to the advancement of computational hardware and the rising interest of the academic community in computationally efficient MPC solutions now it's possible to consider the use of model predictive control algorithms in high sampling rate applications.

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2 EFFICIENT MPC ALGORITHMS

2.1 REQUIREMENTS AND ASSUMPTIONS

The algorithm implemented in a fast sampling application – such as active vibration attenuation has to possess some basic qualities. Just as in any other case of MPC, it has to be able to handle constraints. The stability of the algorithm – invariance is always an important issue, just as the guaranteed feasibility of the process constraints.

It is possible to ensure these properties by deploying hyperellipsoidal or hyperpolyhedral invariant target sets. In the on-line evaluation, the quadratic programming solvers have to be substituted by solvers capable of handling the more advanced problem formulation. Second order cone programming and semidefinite programming solvers demand serious computational cost, thus cannot be used in fast sampling applications.

2.2 POSSIBLE ALGORITHM CANDIDATES

According to our knowledge the highest speed application, in which model predictive control is successfully implemented has a 5 kHz sampling rate. The idea was to use a quadratic programming algorithm which has been re-written directly into machine code. [2] By minimizing the computational overhead and implementing the solver on a digital signal processing board – the use of conventional MPC in vibration attenuation with a bandwidth of 500Hz was possible. The use of a quadratic programming solver implies, that feasibility and invariance of the process is not guaranteed.

Explicit MPC or Multi Parametric MPC is also an area of active research. It takes advantage of the fact, that the active-set QP algorithm is a quadratic program with equality constraints. [7] The offline algorithm determines for all feasible active sets regions, so that the control trajectory of the fixed feedback law will be feasible and optimal. Then the regions are reduced to prevent overlap and duplication. The on-line part performs membership tests to determine in which set the current state lies and performs the associated control law. The advantage is obvious – all possible control laws are determined off-line. The drawback is that for large systems the data storage requirements are enormous, and the search times rise also significantly.

It is possible to develop quadratic programming solvers especially suitable for model predictive control. In conventional MPC approaches the predicted input trajectories are the optimization variables. The state variables are eliminated from the problem – this the number of optimization variables will grow cubically with the horizon length. The matrix factorizations can be replaced by the recursions of state and co-state variables, based on Pontryagin's minimum principle. This way it is possible to develop solvers which have computational complexity only linearly depending on the horizon length. [3] An optimal control problem for input constrained linear systems using the Euler-Lagrange method is developed. For a given set of active constraints, Pontryagin's minimum principle is used to calculate input and state trajectories as functions of initial and terminal states. The number of optimization variables is reduced to plant order, since the multipliers of input constraints are eliminated in the problem. An active set method then successively solved equality constrained problem in a reduced space. Although the above mentioned method can be a magnitude faster than general purpose QP, its numerical robustness is questionable.

There is one more possibility: to use structurally simple but somewhat sub-optimal approaches. If the optimality of the process is not the primary issue, but speed – these methods may be acceptable. The so called Newton Raphson Model (based) Predictive Control algorithm presents a viable way to use MPC in vibration attenuation applications.

3 NEWTON – RAPHSON MPC

As the previous section demonstrated, there are several approaches which ensure the viability of MPC for fast sampling applications. A computationally efficient alternative to the classical predictive control methods can be developed by a unique problem formulation. [4] The so – called Newton – Raphson model predictive control (NRMPC) algorithm uses closed-loop predictions. A pre-stabilized loop is formed, where the predicted performance is not optimized over the system input, rather a new free variable is created. In the absence of constraints the pre-stabilized loop can be optimal in some sense: let's say LQR optimal. The new free variable called perturbation is has a zero value, whenever the constraints are inactive. But during transients it is used to ensure feasibility. It is possible to express the control input at each sampling interval as:

$$u_k = Kx_k + c_k \tag{3.1}$$

where c_{k+i} for $i = 1 \dots n_c$ represents the degrees of design freedom: perturbation. The closed loop system can be expressed then by:

$$x_{k+1} = \Phi x_k + Bc_k \tag{3.2}$$

The dynamics of the system are expressed by the autonomous state-space system:

$$z_{k+1} = \Psi z_k$$
$$\Psi = \begin{pmatrix} \Phi & BE \\ 0 & T \end{pmatrix} \tag{3.3}$$

Vector z includes the state and perturbation vector, $E = [I \quad 0 \quad \dots \quad 0]$ and T is the so called shift matrix, which has the role of shifting the perturbation vector further at each sampling time. The stability of the autonomous system (3.3) is guaranteed by the stability of ψ. An invariant set must exist for this system, which can be defined in the following fashion:

$$E_z = \left\{ z \mid z^T Q_z^{-1} z \le 1 \right\} \tag{3.4}$$

I tis possible to partition Q_z^{-1} similarly to Ψ and get the following blocks: $\hat{Q}_x, \hat{Q}_f, \hat{Q}_{xf}$ and $\hat{Q}_{fx}$. Equation (3.4) may be then expressed by:

$$x^T \hat{Q}_x x \le 1 - f^T \hat{Q}_{fx} x - f^T \hat{Q}_f f \tag{3.5}$$

This inequality is satisfied by all possible z for which $x \in E_x$ if the perturbation vector is not present and providing that $Q_z^{-1} = Q_x^{-1}$. A nonzero perturbation vector may be used to create an invariant ellipsoid, which is larger than the original $x \in E_x$. The maximizer of the right hand side of the equation (3.5) after differentiation is $f = \hat{Q}_f^{-1} \hat{Q}_{fx} x$. If we substitute this back to the inequality:

$$E_{xz} = \{ x \mid x^T Q_{xz}^{-1} x \le 1 \} \tag{3.6}$$

$$Q_{xz} = \left[\hat{Q}_x - \hat{Q}_{xf} \hat{Q}_x^{-1} \hat{Q}_{fx} \right] \tag{3.7}$$

Since $Q_{xz} \le \hat{Q}_x$ and it is clear that $\hat{Q}_x = Q_x^{-1}$ it is obvious that the original invariant set is the subset of the new projection $E_x \subseteq E_{xz}$. It is possible to express matrix Q_{xz} in a more convenient form using a matrix transformation:

$$Q_{xz} = M Q_z T^T, \; x = Mz \tag{3.8}$$

It is difficult to illustrate the augmented invariant set E_z, its projection and intersection with the original state-space. In case the system in question is second order, the augmented state space is already three dimensional with only a one step ahead prediction. Therefore figure 1. attempts to graphically illustrate a merely one dimensional system with a one step ahead prediction – one perturbation. E_z is the augmented invariant set, E_{xz} is the projection into the original state space and E_x the intersection with it.

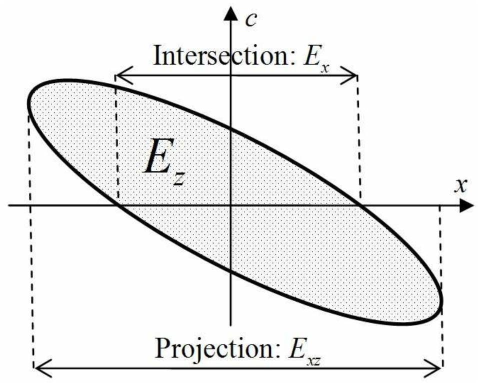

Figure 1 - Augmented ellipsoid and its projection and intersection with the state space

Just as in the case of any other MPC algorithm, invariance and feasibility of the predictions have to be ensured. This is carried out by the aid of the extension of the invariance conditions:

$$\Psi^T Q_z^{-1} \Psi - Q_z^{-1} \leq 0 \tag{3.9}$$

This also may be expressed in a more convenient form for semi-definite programming using Schur complements:

$$\begin{bmatrix} Q_z & Q_z \Psi^T \\ \Psi^T Q_z & Q_z \end{bmatrix} \geq 0 \tag{3.10}$$

In the invariance conditions we need to account for the changes introduced in the calculation of the control input (3.1), namely the presence of the perturbation value. The modified invariance conditions will be therefore:

$$\left\| [K_i^T \quad e_i^T] Q_z^{1/2} \right\| \leq d_i \tag{3.11}$$

Where e_i is the i-th column of the identity matrix and $i = 1, \ldots, p$. We can equivalentrly write for (3.11):

$$d_i^2 - [K_i^T \quad e_i^T] Q_z [K_i^T \quad e_i^T]^T \geq 0 \tag{3.12}$$

The volume of the invariant hyperellipsoid E_{xz} is defined by $\det(TQ_zT^T)$. To maximize the volume while still respecting feasibility and invariance constraints, one has to evaluate the following algorithm:

Algorithm 3.1

Maximize $\log\det(TQ_zT^T)$ by respecting the linear matrix inequalities defining the feasibility (3.12) and invariance conditions (3.10).

The LMI's defining feasibility and invariance do not depend on the current state, this algorithm can be solved off-line. It is assumed that there exists a feedback controller K, a control horizon n_c, a matrix Q_z defining the (projection of) the hyperellipsoid in a way that $x_0 \in E_{xz}$. For a sufficiently large control horizon the feasibility and invariance of the problem can be handled by the degrees of freedom in f. Therefore the perturbation vector I responsible for ensuring that all states will remain in the projection of the augmented invariant set at all times. Furthermore that constraints will not be exceeded but reached if necessary and to use the remaining degrees of freedom to optimize the predicted performance. The controller algorithm may be described as follows:

Algorithm 3.2

Off-line procedure: Calculate a feedback matrix K which is optimal in some sense, while ignoring constraints. Calculate Q_z to maximize the projection of the augmented hyperellipsoid, while respecting conditions for invariance and feasibility.

On-line procedure: At each sampling instant perform the following minimization procedure:

$$\min_f f_k^T f_k \text{ s.t. } z_k^T Q_z^{-1} z_k \le 1 \tag{3.13}$$

Only the first element of the calculated vector f is utilized, then the procedure is repeated at the next sampling instant.

The solution of this minimization problem can be geometrically interpreted as the search for the shortest distance of an ellipsoid from the origin. This is a univariate optimalization problem, graphically illustrated on Figure 2:

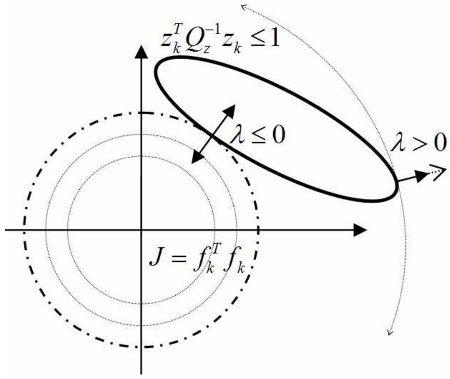

Figure 2 - Graphical illustration of the on-ine optimization problem in NRMPC

If one utilizes the partitioning of Q_z, may rewrite the optimization constraint as:

$$z_k^T Q_z^{-1} z_k = x_k^T \hat{Q}_x x_k + 2 f_k^T \hat{Q}_{fx} x_k + f_k^T \hat{Q}_f f_k \le 1 \tag{3.14}$$

The solution of this optimization problem for the state at a given time will be:

$$f = \lambda M \hat{Q}_{fx} x_k \tag{3.15}$$

$$\Theta(\lambda) = \hat{Q}_{xf} \left[M \hat{Q}_f^{-1} M - \hat{Q}_f^{-1} \right] \hat{Q}_{fx} x_k + x_k^T \hat{Q}_x x_k - 1 = 0 \tag{3.16}$$

Where M is defined as $M = (I - \lambda \hat{Q}_f)^{-1}$ and λ is the unique real root of $\Theta(\lambda)$. The solution of this problem can be found for example by utilizing Newton-Rhapson's algorithm. By using eigenvector-eigenvalue properties of $\hat{Q}_f$ the need to invert general square matrices can be eliminated, instead only element wise inversions are necessary.

While this formulation provides exceptional speed – it is still theoretically suboptimal. QPMPC has still the advantage of using the largest admissible set and providing truly optimal solutions. It is possible to extend NRMPC to match the optimality of QPMPC. [5] There are possible perturbation vectors which satisfy constraints over the prediction horizon and are from the maximal admissible set, which ensures feasibility past the prediction horizon. This and the affine dependence of control input and terminal state imply a set of linear inequalities for the perturbation vector. The vital idea is that scaling of the originally calculated perturbation vector to the additional constraint that the feasibility at the current sample time will imply feasibility at the next one. The extra computation required to obtain the scaling factor can be performed explicitly and does not pose a significant online time increase. While the performance of the algorithm will be practically indistinguishable from QP based MPC, there is a small computational price for that.

Utilizing a nonlinear transformation of variables, it is possible to create a convex formulation for the optimization of the prediction dynamics. [6] This approach leads to a generalization of prediction dynamics used previously, which allows changing the dynamics of the controller depending on the predicted plant state. This method significantly enlarges the volume of the stabilizable set. Since changes take place in the off-line formulation, computational time is not increased.

4 EXPERIMENTAL DEVICE

An experimental device is currently is finished and soon will be ready to test and verify efficient predictive control algorithms like the NRMPC method. The laboratory device consists of a flexible clamped beam manufactured from commercially pure aluminum. This beam is working in cooperation with actuators and sensors to form an active smart structure. The simple structure may for example model the behavior of helicopter rotor blades or solar panels on space structures.

There are piezoelectric transducers permanently bonded to the beam surface. Some of these actuators act as sensors and provide control feedback signal to the algorithm. Others resume actuator mode – thus realizing the control moves calculated by the algorithm.

Figure 3 - Experimental laboratory setup

The beam material, dimension and placement of the piezoelectric actuators were influenced by the results of several finite element simulations of the configuration. Similar existing laboratory setups, practical considerations and other factors also had an effect on the final arrangement.

The beam tip vibration is measured by a high precision LASER distance measuring system. This device is only used for identification and verification purposes, the control signal will be fed to the algorithm from the sensing piezoelectric wafers. Initial controller tests may utilize this signal too.

Initial control algorithm tests will be carried out on a Matlab, xPC target setup. In case the NRMPC algorithm proves to provide the expected computational speeds, further improvements may be expected by using the algorithm on a digital signal processing board.

Currently the electro-mechanical behavior of the beam is mapped. Experimental measurements and also the finite element model of the system will be used to create a state-space mathematical model. This state-space system will then act as a model base in the MPC algorithms. According to the initial tests and measurements a fifteenth order model shall describe the beam’s behavior within the considered bandwidth of 500Hz.

The NRMPC algorithm has been implemented into the Matlab scripting language. Upcoming phases of the work include the transfer this code into C language and the successive creation of S functions.

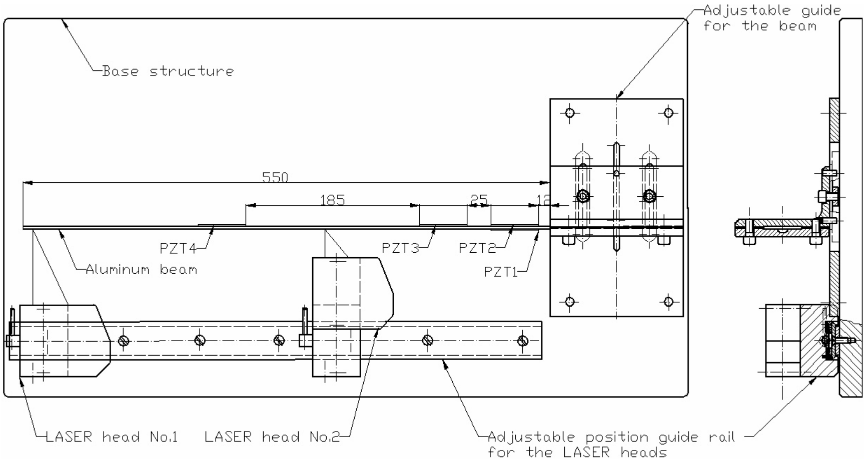

Figure 4 - Clamped active beam with support structure

5 CONCLUSION

NRMPC not only ensures a computationally efficient alternative to the conventional MPC methods with guaranteed invariance and feasibility, but also prevents uncertainty propagating through the predicted states. This control method can be promising in the field of active vibration attenuation.

According to our knowledge NRMPC has not yet been implemented on a physical system. Due to the simple on-line structure of the algorithm, upon the completion of the project we expect to reach computational speeds suitable for active vibration control using typical model orders in the field.

Upcoming phases of the work will implement and test the NRMPC algorithm on a laboratory setup created solely for the purpose of evaluating efficient MPC in vibration attenuation applications.

Acknowledgements

The authors would like to thank the generous financial support of the Slovak Research and Development Agency. The project is under the contract number APVV-0280-06.

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